Integration

After studying differentiation it is natural to study its inverse process. This process is called integration.

Antiderivative

- If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is said to be an anti-derivative or integral of $g(x)$. For example as $\cos x$ is the derivative of $\sin x$, $\sin x$ is anti-derivative of ~~sin x~~ $\cos x$. This fact is symbollically written as $\int \cos x\,dx = \sin x$. The symbol $\int$ (an elongated S) is used to denote the operation of integration and called the integral sign. The function (here $\cos x$) to be integrated is called the integrand; 'dx' denotes the fact that the integration is to be performed with respect to x. (i.e. x is the variable of integration)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + K \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + K$$

$$\int \cos x\,dx = \sin x + K$$

$$\int \sin x\,dx = -\cos x + K$$

$$\int \sec^2 x\,dx = \tan x + K$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + K$$

$$\int \sec x \tan x\,dx = \sec x + K$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + K$$

$$\int e^x\,dx = e^x + K$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + K$$

$$\int \cosh x\,dx = \sinh x + K$$

$$\int \sinh x\,dx = \cosh x + K$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + K$$

$$\int \operatorname{cosech}^2 x\,dx = -\coth x + K$$

$$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + K$$

$$\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x + K$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + K \text{ or } -\cos^{-1}x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + K \text{ or } -\cot^{-1}x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + K \text{ or } -\operatorname{cosec}^{-1}x + c$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + K$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + K.$$

Simple Integration Formule.

ALGEBRA OF INTEGRALS

(i) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

i) $\int \lambda f(x)\,dx = \lambda.\int f(x)\,dx$, for a constant λ.

Examples

(i) $\int (x^6 + x^2 + x + 1)\, dx$

$= \int x^6\, dx + \int x^2\, dx + \int x\, dx + \int 1\, dx$

$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + k$

(ii) $\int 6x^3 (x+5)^2\, dx$

$= \int 6x^3 (x^2 + 10x + 25)\, dx$

$= \int (6x^5 + 60x^4 + 150x^3)\, dx$

$= 6\frac{x^6}{6} + 60\frac{x^5}{5} + 150\frac{x^4}{4} + k$

$= x^6 + 12x^5 + \frac{75}{2}x^4 + K$

Questions

1. (i) $\int 4x^3\,dx$ (ii) $\int x^5\,dx$

(iii) $\int \frac{1}{x\sqrt{x}}\,dx$ (iv) $\int (x+3)(2-x)\,dx$

Questions

(i) $\int \cos x\,dx$ (ii) $\int \frac{\sin x}{\cos^2 x}\,dx$

(iii) $\int \frac{dx}{1-\cos^2 x}$ (iv) $\int \frac{\sin^2 x}{1+\cos x}\,dx$

13.4 INTEGRATION BY SUBSTITUTION.

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution. The integral

$\int f\{g(x)\} g'(x)\, dx$ can be converted to

$\int f(\vartheta)\, d\vartheta$ by substituting $g(x)$ by ϑ, so that if

$\int f(\vartheta)\, d\vartheta = F(\vartheta) + K$, then

$\int f\{g(x)\} g'(x)\, dx = F\{g(x)\} + K$.

This is a direct consequence of chain rule, for

$$\frac{d}{dx}\left[F\{g(x)\} + K\right] = \frac{d}{d\vartheta}\left[F(\vartheta) + K\right] . \frac{d\vartheta}{dx} = f(\vartheta)\frac{d\vartheta}{dx} = f\{g(x)\}\, g'(x).$$

There is no fixed formula for substitution. Keen observation of the form of the integrand will help in choosing the function for which substitution is to be made. However, one must be sure that the derivative of the function so chosen must be present along with dx as in the above case. Occasionally, mere adjustment of a constant may be necessary.

Any symbol for variable viz. s, t, u, v, w, x, y, z may be chosen for substitution other than the variable of the given integral. However, after the integration is over, the original variable should be put back.

Example 2 :

(i) $\int (ax+b)^n\, dx,\ n \neq -1.$

Put $ax + b = \vartheta$, so that $\frac{d\vartheta}{dx} = a$ or $d\vartheta = a dx$.

Hence, $\int (ax+b)^n\, dx = \frac{1}{a}\int (ax+b)^n\, a dx = \frac{1}{a}\int \vartheta^n\, d\vartheta$

$= \frac{1}{a}\frac{\vartheta^{n+1}}{n+1} + C = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$ putting back for ϑ.

(ii) $\int \cos(ax+b)\, dx = \frac{1}{a}\int \cos(ax+b) . a dx$

$= \frac{1}{a}\int \cos\vartheta\, d\vartheta$, putting $ax + b = \vartheta$ and $a\, dx = d\vartheta$

$= \frac{1}{a}\sin\vartheta + C$, where C is an arbitrary constant.

$= \frac{1}{a}\sin(ax+b) + C$, putting back for ϑ.

Similarly,

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + K$$

$$\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a}\sin^{-1}(ax+b) + K, \qquad |ax+b|<1$$

$$\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a}\tan^{-1}(ax+b) + K.$$

(iii) $\int \frac{g'(x)}{g(x)}\,dx = \int \frac{d\vartheta}{\vartheta}$, putting $g(x) = \vartheta$ so that $\frac{d\vartheta}{dx} = g'(x) \Rightarrow g'(x)\,dx = d\vartheta$.

$= ln\,|\vartheta| + C = ln\,|g(x)| + C.$

Taking different functions for $g(x)$, we get

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{a\,dx}{ax+b} = \frac{1}{a}\,ln\,|ax+b| + C.$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{d(\sin x)}{\sin x} = ln\,|\sin x| + C$$

$$\int \tan x\,dx = \int \frac{\sec x\tan x}{\sec x}\,dx = \int \frac{d(\sec x)}{\sec x} = ln\,|\sec x| + C$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx = \int \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}\,dx$$

$$= \int \frac{d(\sec x+\tan x)}{\sec x+\tan x} = ln\,|\sec x+\tan x| + K = ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + K.$$

$$\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx = \int \frac{-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x}{\operatorname{cosec} x-\cot x}\,dx$$

$$= \int \frac{d(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx = ln\,|\operatorname{cosec} x-\cot x| + K = ln\left|\tan\frac{x}{2}\right| + K.$$

(iv) $\int \frac{x^4+4x^3}{x^5+5x^4+7}\,dx = \frac{1}{5}\int \frac{5x^4+20x^3}{x^5+5x^4+7}\,dx$

$= \frac{1}{5}\int \frac{d\vartheta}{\vartheta}$, taking $\vartheta = x^5+5x^4+7$ so that $d\vartheta = (5x^4+20x^3)\,dx = 5(x^4+4x^3)\,dx$

$= \frac{1}{5}\,ln\,|\vartheta| + K$

$= \frac{1}{5}\,ln\,|x^5+5x^4+7| + K.$

(v) $\int \sin^7 x\cos x\,dx = \int \vartheta^7\,d\vartheta$, putting $\sin x = \vartheta$ so that $\cos x\,dx = d\vartheta$.

$= \frac{\vartheta^8}{8} + C = \frac{1}{8}\sin^8 x + C.$

(vi) $\int 2\,e^{\tan^2 x}\tan x\sec^2 x\,dx = \int e^{\vartheta}\,d\vartheta$, putting $\tan^2 x = \vartheta$ so that $2\tan x \cdot \sec^2 x\,dx = d\vartheta$

$= e^{\vartheta} + C = e^{\tan^2 x} + C.$

Integrate the following : (In some cases suggestions have been given for substitution)

1. (i) $\int \sin 3x \, dx$ (ii) $\int \cos ax \, dx$

(iii) $\int \cos (2-7x) \, dx$ (iv) $\int \sin \frac{x}{2} \, dx$

(v) $\int \sec^2 4x \, dx$ (vi) $\int \operatorname{cosec}^2 \frac{x}{3} \, dx$

(vii) $\int \sec (x+2) \tan (x+2) \, dx$ (viii) $\int \operatorname{cosec}\left(x+\frac{\pi}{4}\right) \cot\left(x+\frac{\pi}{4}\right) dx \quad \left(x+\frac{\pi}{4}=z\right)$

(ix) $\int x^2 \cos x^3 \, dx \quad (x^3 = z)$ (x) $\int e^x \sec e^x \tan e^x \, dx \quad (e^x = z)$

(xi) $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx \quad \left(\sqrt{x} = z\right)$.

Integration of Some Trigonometric Functions

If the integrand is of the form $\sin mx \cos nx$, $\sin mx \sin nx$ or $\cos mx \cos nx$, a trigonometric transformation will help to reduce it to the sum of sines or cosines of multiple angles which can be easily integrated.

$$\sin mx \cos nx = \frac{1}{2} \cdot 2 \sin mx \cos nx$$

$$= \frac{1}{2}\left[\sin (m+n)x + \sin (m-n)x\right]$$

$$\sin mx \sin nx = \frac{1}{2}\left[\cos (m-n)x - \cos (m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos (m-n)x + \cos (m+n)x\right]$$

In case where there are more than two factors, succesive transformations will help. For example;

$$\sin mx \cos nx \cos kx = \frac{1}{2} \sin mx \left[\cos (n+k)x + \cos (n-k)x\right]$$

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$$= \frac{1}{2}\left[\sin mx \cos(n+k)x + \sin mx \cos(n-k)x\right]$$

$$= \frac{1}{4}\left[\sin(m+n+k)x + \sin(m-n-k)x \cancel{+\sin} + \sin(m+n-k)x + \sin(m-n+k)x\right]$$

Evaluate (i) $\int \sin 3x \cos 2x\, dx$ (ii) $\int \sin 2x \sin x\, dx$

(iii) $\int \cos 4x \cos 3x\, dx$ (iv) $\int \sin 3x \,.\, \sin 2x \cos 4x\, dx.$

Solution :

(i) $\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int 2 \,.\, \sin 3x \cos 2x\, dx$

$= \frac{1}{2}\int (\sin 5x + \sin x)\, dx = \frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right) + C = -\frac{1}{10}(\cos 5x + 5\cos x) + C.$

(ii) $\int \sin 2x \,.\, \sin x\, dx = \frac{1}{2}\int 2 \sin 2x \sin x\, dx$

$= \frac{1}{2}\int (\cos x - \cos 3x)\, dx = \frac{1}{2}(\sin x - \frac{1}{3}\sin 3x) + C$

$= \frac{1}{6}(3\sin x - \sin 3x) + C.$

[Also $\int \sin 2x \sin x\, dx = \int 2\sin^2 x \cos x\, dx = \int 2z^2 dz$, putting $\sin x = z$

$= \frac{2}{3}z^3 + K = \frac{2}{3}\sin^3 x + K.$

(Verify if the results obtained in the two processes are consistent.)

(iii) $\int \cos 4x \cos 3x\, dx = \frac{1}{2}\int (\cos 7x + \cos x)\, dx = \frac{1}{2}\left(\frac{1}{7}\sin 7x + \sin x\right) + C$

$= \frac{1}{14}(\sin 7x + 7\sin x) + C.$

(iv) $\sin 3x \sin 2x \,.\, \cos 4x = \frac{1}{2}(\cos x - \cos 5x)\cos 4x$

$= \frac{1}{2}(\cos x \cos 4x - \cos 5x \cos 4x) = \frac{1}{4}(2\cos 4x \cos x - 2\cos 5x \cos 4x)$

$= \frac{1}{4}(\cos 5x + \cos 3x - \cos 9x - \cos x)$

So $\int \sin 3x \,.\, \sin 2x \cos 4x\, dx = \frac{1}{4}\int (\cos 5x + \cos 3x - \cos 9x - \cos x)\, dx$

$= \frac{1}{4}\left[\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - \frac{1}{9}\sin 9x - \sin x\right] + C$

$= \frac{1}{180}\left[9\sin 5x + 15\sin 3x - 5\sin 9x - 45\sin x\right] + C.$

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Integrate the following :

1. (i) $\int \sin 4x \cos 3x \, dx$ (ii) $\int \cos 5x \cos 2x \, dx$

(iii) $\int \sin x \cos 4x \, dx$ (iv) $\int \sin 6x \sin 3x \, dx$

(v) $\int \cos 4x \cos 5x \sin 2x \, dx$ (vi) $\int \sin \frac{3x}{4} \cos \frac{x}{2} \, dx$

(vii) $\int \cos 2x \cos \frac{x}{2} \, dx$ (viii) $\int \sin \frac{x}{2} \sin \frac{x}{3} \cos \frac{x}{4} \, dx$.

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION :

The following trigonometric identities can be utilized to simplify certain forms of functions in the integrand with trigonometric substitutions.

$1 - \sin^2\theta = \cos^2\theta$ (or $1 - \cos^2\theta = \sin^2\theta$)

$\tan^2\theta + 1 = \sec^2\theta$ (also $\cot^2\theta + 1 = \text{cosec}^2\theta$)

$\sec^2\theta - 1 = \tan^2\theta$ (also $\text{cosec}^2\theta - 1 = \cot^2\theta$) .

The irrational forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$ can be simplified to radical free functions by putting $x = a \sin\theta$, $x = a \tan\theta$, $x = a \sec\theta$ respectively (or $x = a \cos\theta$, $x = a \cot\theta$, $x = a \text{ cosec}\theta$ respectively). The substitution $x = a \tan\theta$ (or $x = a \cot\theta$) can be useful in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

Exa

Integrate

(i) $\int \frac{dx}{\sqrt{a^2-x^2}}$ (ii) $\int \frac{dx}{x^2+a^2}$

Solⁿ.

(i) Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

and $\theta = \sin^{-1}\frac{x}{a}$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int \frac{\cos\theta}{\cos\theta}\, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\frac{x}{a} + C$$ (Ans.

(ii) Let $x = a\tan\theta$, $dx = a\sec^2\theta \, d\theta$

and $\theta = \tan^{-1}\frac{x}{a}$

$$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta \, d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{a^2\sec^2\theta}$$

$$= \int \frac{1}{a} d\theta$$

$$= \frac{\theta}{a} + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$ (Ans.

Standard formulae

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

4. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + A$
$= \sinh^{-1}\frac{x}{a} + B$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + A$
$= \cosh^{-1}\frac{x}{a} + B$

Questions

(i) $\int \frac{dx}{\sqrt{11-4x^2}}$ (ii) $\int \frac{\cos\theta}{\sqrt{4-\sin^2\theta}} d\theta$

(iii) $\int \frac{x^2}{\sqrt{36-x^6}} dx$ $(x^3=z)$ (iv) $\int \frac{x+3}{\sqrt{9-x^2}} dx$

Integration By Parts:

If v and w are differentiable functions of x, then

$$\frac{d}{dx}(vw) = v\frac{dw}{dx} + w\frac{dv}{dx}$$

or $$v\frac{dw}{dx} = \frac{d}{dx}(vw) - w\frac{dv}{dx}$$

Integrating both sides,

$$\int v\frac{dw}{dx}dx = \int \frac{d}{dx}(vw)\,dx - \int w\frac{dv}{dx}dx$$

$$= wv - \int w\frac{dv}{dx}dx$$

Setting $\frac{dw}{dx} = u$, $w = \int u\,dx$, the result can be rewritten as

$$\int uv\,dx = \left(\int u\,dx\right) \times v - \int\left(\left(\int u\,dx\right) \times \frac{dv}{dx}\right)dx$$

This rule is called the rule of 'integration by parts'.

In words, Integral of the product of two functions

= (Integral of first function) × second function

− Integral of (integral of first × derivative of the second)

The following table gives a proper choice of 'first' and 'second' functions in certain cases. Here $m \in N$, n may be zero or any positive integer.

Functions to be integrated	First function	Second Function
$x^n e^x$	e^x	x^n
$x^n \cos x$	$\cos x$	x^n
$x^n \sin x$	$\sin x$	x^n
$x^n (\ln x)^m$	x^n	$(\ln x)^m$
$x^n \sin^{-1}x$	x^n	$\sin^{-1}x$
$x^n \cos^{-1}x$	x^n	$\cos^{-1}x$
$x^n \tan^{-1}x$	x^n	$\tan^{-1}x$

Usually among exponential function (E), trigonometric function (T), algebraic function (L) and inverse trigonometric function (I), the choice of 'first' and 'second' function is made in the order ETALI.

Exa: Integrate

(i) $\int x\cos x \, dx$

Sol:

$\int x\cos x \, dx$

$= \left(\int \cos x \, dx\right) \cdot x - \int \left(\left(\int \cos x \, dx\right) \times \frac{dx}{dx}\right) dx$

$= x\sin x - \int \sin x \cdot 1 \, dx$

$= x\sin x + \cos x + C$

(iii) $\int \tan^{-1} x \, dx$

$= \int 1 \cdot \tan^{-1} x \, dx$ (here '1' is the first function)

$= x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2} dx.$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx.$

$= x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + C$

When $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. or $\log x$ is present alone in the integrand, $1 = x^0$ has to be taken as the first function.

Questions

1. (i) $\int x^3 e^{x} dx$ (ii) $\int x \cos^2 x \, dx$
(iii) $\int x^2 \sin x \, dx$ (iv) $\int x \tan^2 x \, dx$

Definite Integral

Fundamental theorem of Integral calculus.

Statement:

If $f(x)$ is continuous in the interval $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

In case of definite integrals,

(i). $\int_a^b [g(x) + h(x)] \, dx = \int_a^b g(x) dx + \int_a^b h(x) dx$

and

(ii) $\int_a^b \lambda g(x) \, dx = \lambda \int_a^b g(x) dx$

Examples:

(i) $\int_1^2 x^3 dx = \left|\frac{x^4}{4}\right|_1^2 = \left(\frac{x^4}{4}\right)_{x=2} - \left(\frac{x^4}{4}\right)_{x=1}$

$= \frac{2^4}{4} - \frac{1^4}{4}$

$= 4 - \frac{1}{4}$

$= \frac{15}{4}$

(ii) $\int_0^{\pi/2} \sin x \, dx = \left|-\cos x\right|_0^{\pi/2} = \left(-\cos\frac{\pi}{2}\right) - (-\cos 0)$

$= 0 - (-1)$

$= 1$

(iii) $\int_0^{\pi/2} x \cos x \, dx$

We first find the indefinite integral

$\int x \cos x \, dx = \sin x \cdot x - \int \sin x \cdot 1 \cdot dx$

$= x \sin x + \cos x + K$

$\therefore \int_0^{\pi/2} x \cos x \, dx = \left|x \sin x + \cos x\right|_0^{\pi/2}$

$= \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 \cdot \sin 0 + \cos 0)$

$= \frac{\pi}{2} - 1$

Elementary Properties of Definite Integrals.

(i) $\int_a^b f(x)\,dx = \cancel{\int_b^a f(x)\,dx} - \int_b^a f(x)\,dx$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

i.e. definite integral is independent of the symbol of variable of integration.

(iii) $\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$,

$a < d < b$.

Questions

Evaluate the following integrals

1. (i) $\int_1^3 \frac{dx}{x^3}\,dx$ (ii) $\int_0^{\pi/2} (\cos x - \sin x)\,dx$

(iii) $\int_{-1}^{2} (2x+1)^2 (x-2)\,dx$ (iv) $\int_0^{\pi/4} \tan^2 x\,dx$

(v) $\int_0^2 x^2 e^{x^3}\,dx$

Some more Properties of Definite Integrals

(i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(ii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even function} \\ 0 & \text{if } f \text{ is odd function} \end{cases}$

(iii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

* For an even function $f(-x) = f(x)$ and for an odd function $f(-x) = -f(x)$

Exa: $\int_{-\pi/4}^{\pi/4} \cos^3 x\,dx = 2\int_0^{\pi/4} \cos^3 x\,dx$ (as $\cos^3 x$ is an even function)

$= 2\int_0^{\pi/4} (1-\sin^2 x)\cos x\,dx$

$= 2\int_0^{1/\sqrt{2}} (1-z^2)\,dz$ Putting $\sin x = z$

$= 2\left|z - \frac{z^3}{3}\right|_0^{\frac{1}{\sqrt{2}}}$

$= 2\left(\frac{1}{\sqrt{2}} - \frac{1}{3\cdot 2\sqrt{2}}\right)$

$= \frac{2}{\sqrt{2}}\left(1-\frac{1}{6}\right)$

$= \frac{5}{6}\sqrt{2}$

Questions

Evaluate the following integrals

(i) $\int_0^{\pi/2} \frac{dx}{1+\tan x}$ (ii) $\int_{-\pi/4}^{\pi/4} \cos^2 x \, dx$

(iii) $\int_0^{\pi} \sin x \cdot \cos^2 x \, dx$

Area Under Plane Curves:

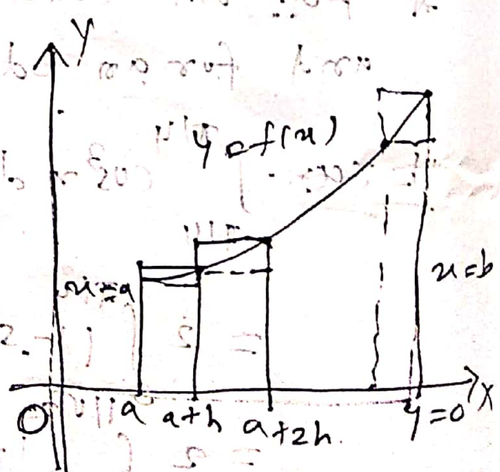
The definite integral was defined as the limit of a sum. If the limit exist, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh)$$

$$= \lim_{n\to\infty} h \sum_{r=1}^{n} f(a+rh)$$

where $h = \frac{b-a}{n}$

Suppose $f(x)$ is positive and increasing in $[a,b]$. Let the ordinates corresponding to $x = a + rh$ $(r = 0, 1, 2, \ldots, n)$ be drawn



and the inner rectangles (rectangles drawn in the figure just below the curve) and outer rectangles (rectangles drawn just covering the curve; each of width h be completed.

Then $S_1 = h\sum_{r=0}^{n-1} f(a+rh)$ is the sum of the areas $hf(a)$, $hf(a+h)$, ---- $hf(a+rh)$ of the inner rectangles. Also, $S_2 = h\sum_{r=1}^{n} f(a+rh)$ is the sum of the areas $hf(a+h)$, ---- $hf(a+rh)$ of the outer rectangles.

The actual area A under the curve $y=f(x)$, above the x-axis and betn the ordinates at $x=a$ and $x=b$ lies betn S_1 & S_2 i.e. $S_1 < A < S_2$. As $n\to\infty$, $h\to 0$, the difference betn S_1 (or S_2) and A is reduced and ultimately S_1 (or S_2) approaches A.

Then $A = \lim\limits_{n\to\infty} S_1 = \lim\limits_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$

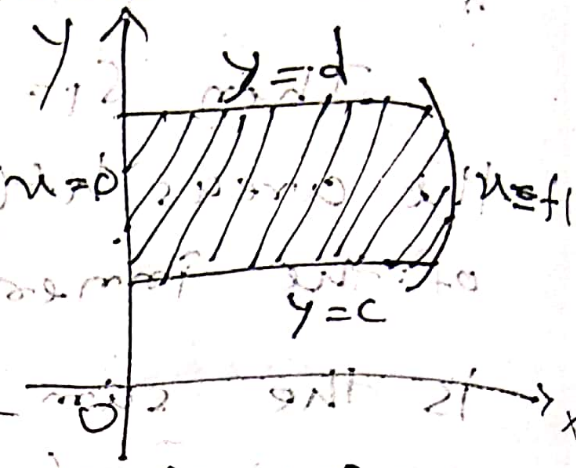
$= \lim\limits_{n\to\infty} S_2 = \lim\limits_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$

i.e. $A = \int_a^b f(x)\,dx$

Thus the definite integral $\int^b f(x)\,dx$ represents the area under the curve $y=f(x)$ above the x-axis and betn the ordinates $x=a$ and $x=b$.

If $f(x)$ is decreasing in $[a,b]$, then S_1 is the sum of the areas of the outer rectangles and S_2 that of the inner rectangles and $S_1 > A > S_2$. But $\lim\limits_{n\to\infty} S_1 = \lim\limits_{n\to\infty} S_2 = A$ & the same result follows.

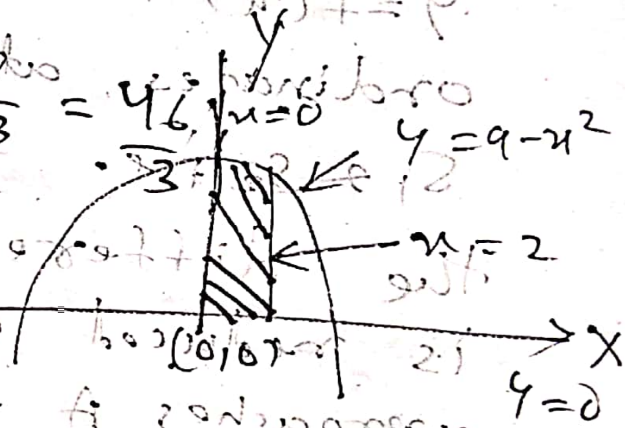
Area between the curve $x=f(y)$, $x=0$ and the abscissa $y=c$, $y=d$ can be similarly A shown to be $\int_c^d f(y)\,dy$.



Exa (i) Area of the region enclosed by $y = 9 - x^2$, $y = 0$ and the ordinates $x = 0$ and $x = 2$ is given by

$$A = \int_0^2 (9 - x^2)\,dx$$

$$= \left|9x - \frac{x^3}{3}\right|_0^2 = 18 - \frac{8}{3} = \frac{46}{3}$$



Questions

Find the area bounded by

(i) $y = e^x$, $y = 0$, $x = 4$, $x = 2$

(ii) $y = x^2$, $y = 0$, $x = 1$

(iii) $y = \sin x$, $y = 0$, $x = \frac{\pi}{2}$